DIFFERENTIAL GEOMETRIC APPROACH TO THE SINGULAR PROBLEMS OF OPTIMAL CONTROL

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Abstract

We consider optimal control problems on manifolds which are linear in the control. Some results of the type called "highway theorems" are obtained. Our theory is used both to provide theoretical results, and to construct a synthesis on the space of representation of the orthogonal group.

1. Introduction.

The use of techniques developed for exterior differential systems in the calculus of variations and optimal control theory seems quite promising. In the nonsingular case this approach leads to differential forms on the jet manifold [1]. We consider the singular case when the problem is linear in the control; this enables us to obtain much more effective results using differential forms defined on the original manifold.

2. Problem formulation.

Let's consider a control problem of the following kind which is linear in the control:

Problem 1. To minimize

$$\int_0^T f_0(x(t), v(t))dt$$

subject to the restrictions

$$\frac{dx_i}{dt} = f_i(x(t), v(t)), \quad (i = 1, ..., n);$$
$$v(t) \in V; \quad x(\cdot) \in PC^1([0,T], \mathbf{R}^n);$$

and the boundary conditions

$$x(0) = a, \; x(T) = b.$$

Here the functions $f_i(x, v)$ are linear in v and smooth in x; V is a polyhedron from $\mathbf{R}^k$; $PC^1([0,T], \mathbf{R}^n)$ stands for the space of piecewise smooth mappings of the segment $[0,T]$ into $\mathbf{R}^n$ equipped with the norm of a subspace of $C([0,T], \mathbf{R}^n)$.

It was shown in [2] that we can reduce *Problem 1* to that of minimization of a curvelinear integral. Let us consider the more general problem:

Problem 2.

$$J(x(\cdot)) = \int\limits_{x(t)} \Omega \rightarrow inf$$

where

$$\begin{aligned} &x'(t) \in K(x(t)) \text{ a.e. } t \in [0,1] \\ &x(0) = a, \; x(1) = b \end{aligned} \tag{1}$$

The phase space X is an n-dimensional smooth manifold; Ω is differential 1-form on X; $K(x)$ is a cone from the tangent plane $T_x X$ at each point of the manifold X. As trajectories we denote piecewise smooth mappings from the segment $[0,1]$ to X, i.e. they belong to the space $PC^1([0,1], X)$ with topology induced from $C([0,1], X)$.

We vary the initial point $a \in X$ and fix the target point $b \in X$, i.e. we want to construct a synthesis of optimal trajectories in our problem. A set of trajectories satisfying the condition (1) we denote ***admissible***

Let us assume that a compact Lie group acts on the phase space X and retain the invariant cone K and the form Ω. It is natural to suppose this, because a lot of problems of mechanics and physics can be formulated with the help of the terminology of Hamiltonian systems on Lie algebras. More important seems the case when X is a linear space, i.e. the space of representations of the group G.

3. Definitions and main results.

Let the group G act on the manifold X. This means that we have a homomorphism of G into the group of all automorphisms of X. Denote by A_g the mapping corresponding to the element g from G, by $(A_g)^{\star}(x)$ its differential at point $x \in X$, and by $(A_g) \star (x)$ the adjoint operator of $(A_g)^\star(x)$; it maps from the space $T^\star_{A_gx}X$ of differential forms at the point A_gx to this one $T^\star_xX$ at point x; $K(x) \in T_xX$, $\Omega(x) \in T^\star_xX$.

Definition. Let's call *Problem 1 equivariant* with respect to the group G if G retains the form Ω and the cone field K. This means that for any $g \in G$

$$(A_g)^\star(x)[K(x)] = K(A_gx)$$
$$(A_g) \star (x)[\Omega(A_gx)] = \Omega(x)$$

Definition. If Y is a submanifold of X, denote for any point $y \in Y$

$$K_Y(y) = K(y) \cap T_yY$$

It is a cone in the tangent bundle of the manifold Y. Denote by Ω_Y the restriction of the form Ω to the manifold Y. Suppose also that a, b belong to Y. Now we can formulate

Problem 3.

$$\int_{y(t)} \Omega_Y \to \inf$$

subject to the restrictions

$$y'(t) \in K_Y(y(t)) \text{ a.e. } t \in [0,1]$$
$$y(0) = a$$
$$y(1) = b$$

Let's call *Problem 3 a restriction* of *Problem 2* to the manifold Y.

Definition. The manifold Y is said to be *completely extremal* if an extremal *arc* of *Problem 3* with any boundary conditions is also an optimal *arc* of *Problem 2* with the same boundary conditions.

Let the point b be stable relatively to the action of group G.

Definition. A completely extremal manifold Y is called *generating* for the optimal synthesis in *Problem 2* if for any optimal *arc* $x(t)$ of it there exist an optimal *arc* $y(t)$ of problem 3 and an element $g \in G$ such that

$$A_gy(t) = x(t), \quad t \in [0,1]$$

Suppose that X is a linear space.

Definition. The differential form Ω is called *convex* with respect to the cone $K(x)$ if for any fixed points $\xi, \eta \in X$ the functional

$$J(x(\cdot)) = \int_{X(t)} \Omega$$

is convex on the space of functions $x(t)$ such that

$$x'(t) \in K(x(t))$$
$$x(0) = \xi$$
$$x(1) = \eta$$

If J is strictly convex, then Ω is a strictly convex differential form.

Lemma 1. *Let the differential form Ω be completely integrable, i.e. $\Omega = P(x)dQ(x)$. Suppose that the level surfaces of $Q(x)$ are strictly supporting planes to the cone $K(x)$ and that the restrictions of the function $P(x)$ to these planes are convex functions. Then Ω is convex.*

Proof. Transform the coordinate system such that the planes $Q(x) = C$ coincide with the coordinate planes $z_1 = C$. Within the new coordinate system the function P looks like $P(z_1, z_2, ..., z_n)$ and is convex with respect to the variables $z_2, ..., z_n$ for any fixed z_1. Since the planes $z_1 = C$ are strictly supporting to $K(x)$, the coordinate z_1 strictly increases along any admissible trajectory. Subsequently we can take z_1 as a new independent variable in the integral $\int \Omega$

$$J(z(\cdot)) = \int\limits_{z(0)}^{z(1)} P(z_1, z_2(z_1), ... z_n(z_1))dz_1$$

From the corresponding inequality for the function P it is easy to deduce this one

$$J(\alpha z(\cdot) + (1-\alpha)z^{\star}(\star)) \leq \alpha J(z(\cdot)) + (1-\alpha)J(z^{\star}(\cdot)) \qquad (0 \leq \alpha \leq 1).$$

QED

Suppose that the polyhedral cone $K(x)$ from *Problem 2* has generatrices $\lambda_1(x), ... \lambda_r(x)$ which are smooth, linearly independent vector fields on X. Fix the following notation: N is a subset of the set $\{1, ...r\}$; by the face L of the cone K determined by generatrices $\lambda_i(x)(i \in N)$ we mean the set $Con\{\lambda_i(x), i \in N\} = \{\Sigma_{i \in N} m_i \lambda_i(x), m_i \geq 0\}$; by the relative interior of the face L (riL) we denote the set $\{\Sigma_{i\in N} m_i \lambda_i(x), m_i > 0\}$. We call the trajectory $x(t)$ *L-singular* on a segment (t_0, t_1) if $x'(t) \in riL(x(t)) \; \forall \, t \in (t_0, t_1)$.

Theorem 1. *Suppose that the distribution of subspaces determined by a face L of K is integrable and that $x(t)$ is L-singular on (t_0, t_1). Then a necessary condition for a path $x(t)$ to be optimal in Problem 2 is*

$$i_L^\star d\Omega_{x(t)} = 0 \quad \forall \, t \in (t_0, t_1).$$

Here $i_L^\star$ is the operator of restriction of the differential forms to the face L. The proof can be found in [2].

Let the face L_1 of the cone K be determined by the vectors $\lambda_1(x), \ldots, \lambda_k(x)$; and the face L_2 by $\lambda_{k+1}(x), \ldots, \lambda_{k+m}(x)$.

Theorem 2. *Suppose that the distribution of planes, determined by the faces of the cone $Con\{\lambda_i(x), 1 \leq i \leq k+m\}$, are integrable and that at a point $q \in X$*

$$d\Omega_h\{\lambda_i(x), \lambda_j(x)\} < 0 \quad (i \leq k, j \geq k+1) \tag{2}$$

Suppose that trajectory $x(t)$ $\;(x(t_0) = h)$ at the point q switches from the regime corresponding to face L_2 to the one corresponding to face L_1. Then $x(t)$ is non-optimal.

Proof. Since $x(t) \in PC^1([0,1])$ is switching from L_2 to L_1, there exists $s > 0$ such that

$$x'(t) = \sum_{j=k+1}^{k+m} u_j(t)\lambda_j(x(t)) \quad t_0 - s < t < t_0, \tag{3}$$

$$x'(t) = \sum_{i=1}^{k} u_i(t)\lambda_i(x(t)) \quad t_0 < t < t_0 + s, \tag{4}$$

where

$$u_j(t) \quad (j \geq k+1) \text{ are continuous on } [t_0 - s, t_0];$$
$$u_i(t) \quad (i \leq k) \text{ are continuous on } [t_0, t_0 + s]$$

and

$$u_q(t) \geq 0 \quad (1 \leq q \leq k+m); \tag{5}$$

$$\sum_{j=k+1}^{k+m} u_j(t_0 - 0) > 0; \tag{6}$$

$$\sum_{i=1}^{k} u_i(t_0 + 0) > 0. \tag{7}$$

Through the point q we hold a leaf M of foliation corresponding to the distributions of planes $Lin\{\lambda_1, ..., \lambda_{k+m}\}$. All the following constructions will be built inside the leaf M. Let's change our coordinate system to replace our cone K_M at some neighbourhood O of q by the first orthant. Fix an index p and consider the foliation corresponding to the distribution $Lin\{\lambda_1, ..., \lambda_{p-1}, \lambda_{p+1}, ...\lambda_{k+m}\}$. In the neighbourhood O its leaves are determined by the equations $F_p(x) = C$, where $F_p(x)$ is a smooth function with non-vanishing gradient. Thus $\forall x \in O$

$$< \tfrac{\partial F_p}{\partial x}, \lambda_q >= 0 \text{ if } p \neq q, \tag{8}$$

$$< \tfrac{\partial F_p}{\partial x}, \lambda_p >= S_p(x) \neq 0 \tag{9}$$

Without loss of generality we can consider that $S_p(x) > 0$ for any x from O. Let us study the mapping $F: \ O \to \mathbf{R}^{k+m}$, determined by the formulas $z_p = F_p(x)(1 \leq p \leq k+m)$. From the linear independence of the vectors $l_i(x)$ and the expressions (8)-(9), it is easy to deduce that $\partial F_p / \partial x$ are linearly independent too, i.e. the mapping F determines local coordinates on M in the neighbourhood O. Simultaneously, $S_p(x)$ becomes a function with respect to z, and we denote it by $Q_p(x)$. When moving along the admissible trajectories of *Problem 2* inside the neighbourhood O, the following equation is valid

$$z'_p =< \frac{\partial F_p}{\partial x}, \sum_{i=1}^{k+m} u_i \lambda_i >= u_p S_p(x) = u_p Q_p(z) \tag{10}$$

The right sides of (10) for various $(u_1, ...u_{k+m})$, $(u_p \geq 0)$, fill the first orthant. This means that within the new coordinates the cone $K_M(z)$ coincides with the first orthant; admissible arcs will be vector-functions $z(t)$ with non-decreasing coordinates. The system (3)-(4) for the image $z(t)$ of the trajectory x(t) become like this

$$\begin{aligned} &z'_i(t) = 0 \ (i \leq k) && \text{for } t_0 - s < t <_0 \\ &z'_j(t) = 0 \ (j \leq k+1) && \text{for } t_0 < t < t_0 + s \end{aligned}$$

Define a variation $\underline{z}(t, \delta)(\delta < s)$ of the trajectory $z(t)$ as below:

$$\begin{aligned} &\underline{z}(t, \delta) = z(t) && \text{for } t \notin (t_0, t_0 + s) \\ &\underline{z}_i = z_i(t+\delta) \quad (i \leq k); \quad \underline{z}_j(t, \delta) = z_j(t_0 - \delta) \quad (j \geq k+1) && \text{for } t_0 - \delta < t < t_0, \\ &\underline{z}_i(t, \delta) = z_i(t_0 + \delta) \quad (i \leq k); \ \underline{z}_j(t, \delta) = z_j(t - \delta) \quad (j \geq k+1) && \text{for } t_0 < t < t_0 + \delta. \end{aligned}$$

The variation $\underline{x}(t, \delta)$ of the trajectory $x(t)$ will be inverse image $F^{-1}(\underline{z}(t, \delta))$ of the trajectory $\underline{x}(t,\delta)$. Calculating the increment of our functional with this variation:

$$\Delta J = \int\limits_{\underline{x}(t,\delta)} \Omega - \int\limits_{x(t)} \Omega = \int\limits_{\pi} \Omega,$$

where π is the closed curve, formed by the trajectory, which from the point $x(t-\delta)$ moves at first along system (4), then along system (3) up to the point $x(t_0+\delta)$ and finally returns along the trajectory $x(t)$ down to the point $x(t_0-\delta)$. It is easy to see, that

$$\Delta J = \int\limits_{\pi} \Omega = \int\int d\Omega =$$

$$= C\delta^2 d\Omega_q\{\sum_{i=1}^{k} u_i(t_0-0)\lambda_i, \sum_{j=k+1}^{k+m} u_j(t_0+0)\lambda_i\} + o(\delta^2)$$

where C is some positive constant. The coefficient of δ^2 is negative from (2),(5) and (6). Consequently if we take a sufficiently small $\delta > 0$, we obtain that $\Delta J < 0$ and the trajectory $x(t)$ is non-optimal.

QED

If the conditions of *Theorem 2* are valid we shall say that face L_1 *majorizes* face L_2.

4. Applications to problems on spaces of representation of Lie groups.

Consider G - a semisimple compact Lie group - and its Lie algebra g. Denote its adjoint representation $Ad_G: G \to GL(g)$. Suppose that we have a cone K from g invariant with respect to Ad_G, and also an invariant differential form Ω which is convex with respect to K. Formulate *Problem 2* where g is the phase space, Ω is the differential form and K is the cone. Let h be the Cartan subalgebra of g. Well need the following theorem:

Theorem 3. *The subalgebra h is a completely extremal manifold. If Ω is strictly convex and the point $x(1) = b$ remains stable with respect to the action of Ad_G, then h is generating.*

We'll consider one special case, where G is the orthogonal group $O(n)$, i.e. the Lie group of orthogonal transformations of space $\mathbf{R}^n$. Its Lie algebra $o(n)$ - the algebra of symmetric $(n \times n)$-matrices in $\mathbf{R}^n$ - can be considered as a linear space of quadratic forms of n variables. At each point $x \in o(n)$ we take the cone K_+ of non-negative definite quadratic forms, which is obviously invariant with respect to the action of $O(n)$. Matrices with non-negative definite derivatives will be called admissible. Let's formulate Problem 2 with $o(n)$ as a phase space, K_+ as a cone and with the differential form

$$\Omega = p_2(x) dp_1(x)$$

where $p_1(x)$ is first coefficient of the characteristic polynomial of the matrix x (i.e. minus trace of x), and $p_2(x)$ is the second one. It is well known that this form is invariant with respect to the action of $O(n)$. For convenience rewrite

$$\Omega = -p_2(x) d(-p_1(x))$$

Lemma 2. *The planes $\pi_C = \{p_1(x) = C\}$ are strictly supporting to K_+.*

Proof.

$$\frac{d}{dt} Tr x'(t) = \sum_{i=1}^{n} x'_{ii}(t).$$

Moreover the matrix $x' \geq 0$, consequently $x'_{ii} \geq 0$ for all i and $\frac{d}{dt} Tr(x'(t)) \geq 0$, and equality is fulfilled only for the matrix $x'(t) = 0$. Thus for all $x' \in K_+$, $x \neq 0$ we have $d(-p_1(x)) > 0$.

QED

Lemma 3. *The restriction of the function $-p_2(x)$ to the plane π_C is a strictly convex function.*

Proof. By definition

$$p_2(x) = \sum_{i \neq j} det \begin{pmatrix} x_{ii} & x_{ij} \\ x_{ji} & x_{jj} \end{pmatrix}) \quad = \sum_{i \neq j} (x_{ii} x_{ij} - x_{ij}^2) \tag{11}$$

Since the function $\sum_{i \neq j} x_{ij}^2$ is convex, we have to justify only the convexity of the restriction of the quadratic form $W(x) = -\sum_{i \neq j} x_{ii} x_{jj}$ to the plane

$$\pi_C = \{\sum x_{ii} = C\}.$$

The form $W(x)$ depends only on the diagonal entries x_{ii} of the matrix x. In the space T, the matrix W of the quadratic form $W(x)$ will be like this:

$$W = \begin{pmatrix} 0 & -1 & -1 & \cdots & -1 \\ -1 & 0 & -1 & \cdots & -1 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ -1 & -1 & -1 & \cdots & - \end{pmatrix}$$

It is easy to see that the matrix W has 1 as an eigenvalue of mutiplicity $(n-1)$, and $(1-n)$ as an eigenvalue of multiplicity 1, corresponding to the eigenvector $e = (1, ..., 1)$ and that e is a normal vector (in T) to the plane $\pi_C \cap T$. Consequently the matrix W has the plane $\pi_C \cap T$ as an invariant subspace corresponding to the eigenvalue 1, and thus $W(x)|_{\pi c \cap T}$ is a strictly convex function. From (11) and because of dimensionality reasons, we obtain that the function $(-p_2(x))|_{\pi c \cap T}$ is strictly convex too.

From *Lemmas 1,2* and *3* we can deduce that the form Ω is strictly convex with respect to the cone K_+. The Cartan subalgebra h of the algebra $o(n)$ consists of matrices which can be simultaneously diagonalized. The manifold h, according to *Theorem 3*, is completely extremal. For the boundary condition $x(1) = \Psi I$, where I is the unit matrix and y is constant, h is a generating manifold. We'll build a synthesis of optimal trajectories on the manifold h in the basis where h consists of diagonal matrices. By the help of the mapping $x \to diag(x)$ we can replace the phase space of *Problem 3* by $\mathbf{R}^n$. Consequently, the cone $K_h = \mathbf{R}^n_+$, and the differential form becomes

$$\Omega_h = -\sum_{k \neq m} x_k x_m d(\sum_{j=1}^{n} x_j)$$

where the matrix is

$$x = \begin{pmatrix} x_1 & 0 & \cdots & 0 \\ 0 & x_2 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & \cdot & \cdots & x_n \end{pmatrix}$$

Non-singular extremals correspond to the generatrices of the cone $\mathbf{R}^n_+$. It means that along the non-singular extremals only one coordinate of x increases.

Let's construct the synthesis of singular extremals. Take the face L of the cone $\mathbf{R}^n_+$ determined by the basis vectors with numbers from the set

$$\beta = \{j_1, ..., j_k\} \in \{1, ..., n\}$$

Let the trajectory $x(t)$ be optimal, then

$$d\Omega_h = \sum (x_k - x_m) dx_k \wedge dx_{m'}$$

and the restriction

$$i_L^\star d\Omega_h = \sum_{k,m\in\beta} (x_k - x_m) dx_k \wedge dx_m. \qquad (12)$$

According to *Theorem 1*

$$i_L^\star d\Omega_h = 0,$$

consequently

$$x_{j1} = x_{j2} = ... = x_{jk}.$$

This means that along singular segments of the optimal trajectory, only one *multiple* root of characteristic polynomial of the matrix x increases. Now well show that this increasing root must be minimal. At any point $x \in \mathbf{R}^n$ consider a set of indices

$$\beta(x) = \{1 \le j \le n | x_j = min_k x_k\},$$

which corresponds to minimal entries of the vector x. Let $L_{min}(x)$ denote the corresponding face of the cone $\mathbf{R}_+^n$. It is easy to see from (12) that the face $L_{min}(x)$ majorizes all the other faces of the cone $\mathbf{R}_+^n$. With Theorem 2 it means that switching from the regime of increasing a non-minimal root to the regime of increasing a minimal root is non-optimal for any $x \in \mathbf{R}^n$. From $x(t) \in PC^1([0,1])$ we can deduce that there exists a partition $0 = t_0 < t_1 < ... < t_N = 1$ of the segment $[0,1]$ such that on any subinterval only one (single or multiple) characteristic root of the matrix x increases. If on some subinterval a non-minimal root increases, we can not reach the target point ΨI, because on the rest of the trajectory the minimal root remains constant. By direct justification of Bellmans conditions it is easy to prove that this synthesis on h is optimal and we omit the proof.

According to *Theorem 3* we can obtain the whole synthesis on the manifold $o(n)$ by using the action of group $O(n)$ on the constructed one.

We consider this theory as an original development of Gohs ideas ([3], [4]). But unlike his works the results presented here have an invariant form and so give a possibility to apply them to different problems which can be expressed in a coordinate free form, for instance problems of stochastic optimal control.

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